Micro module 16: Game theory: practice problems

*Practice problems and illustrative test questions for the final exam*

(The attached PDF file has better formatting.)

This posting gives sample final exam problems. Other topics from the textbook are asked as well; these problems are just examples. All final exam problems are multiple choice; some practice problems are not multiple choice so that the solutions can be better explained.

Exercise 16.1: Prisoner’s dilemma

<table>
<thead>
<tr>
<th>Rachel’s Strategy</th>
<th>Confess</th>
<th>Not Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>Jacob gets 5 years</td>
<td>Jacob gets 8 years</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 5 years</td>
<td>Rachel gets 1 year</td>
</tr>
<tr>
<td>Not Confess</td>
<td>Jacob gets 1 year</td>
<td>Jacob gets 2 years</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 8 years</td>
<td>Rachel gets 2 years</td>
</tr>
</tbody>
</table>

A prisoner’s dilemma is a game in which the players have a Pareto optimal outcome which is the better outcome for both players, but each player has a dominant strategy leading to an outcome that is not Pareto optimal.

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

**Part A:** Know the procedure to determine if a player has a dominant strategy.

Each player has two options. Games need not be symmetric, so one player may have a dominant strategy and the other player may not. We label Jacob’s strategies *right* and *left* and Rachel’s strategies *up* and *down.* In the prisoner’s dilemma, the two strategies are labeled *confess* and *non-confess.*

To see if Jacob has a dominant strategy, ask:

- If Rachel chooses *not-confess*, Jacob chooses *confess*, since it leads to fewer years of prison: 1 year vs 2 years.
- If Rachel chooses *confess*, Jacob chooses *confess*, since it leads to fewer years of prison 5 years vs 8 years.

Most games seek to maximize a variable (usually wealth). In the prisoner’s dilemma, consider years of prison as negative wealth. Minimizing years of prison is like maximizing wealth.
Part B: The prisoner’s dilemma is symmetric, so Rachel’s dominant strategy is the same as Jacob’s dominant strategy.

Part C: Know the solution method and a few general rules.

For each outcome, ask: “If Jacob and Rachel are in that outcome, and Rachel will persist in her choice, would Jacob prefer another choice? If Jacob will persist in his choice, would Rachel prefer another choice?” If both answers are No, the outcome is a Nash equilibrium.

- A game may have zero, one, or more than one Nash equilibria.
- If each player has a dominant strategy, the outcome of the dominant strategies is a Nash equilibrium and it is the only Nash equilibrium.

In the prisoner’s dilemma, each player has a dominant strategy, so there is a unique Nash equilibrium: Jacob confesses and Rachel confesses.

Part D: Know the solution method. For each outcome, ask: “Is there any other outcome that is better for one player and at least as good for each other player?” If the answer is No, the outcome is Pareto optimal.

A Nash equilibrium may or may not be Pareto optimal. In the prisoner’s dilemma, confess/confess is the only Nash equilibrium but it is not Pareto optimal. The other three outcomes are not Nash equilibria, but they are all Pareto optimal. (Pareto optimal is a poor term, since it implies that the outcome is the best possible, and only one outcome can be the best possible.)

Question: If an outcome is the only Nash equilibrium, how can it not be Pareto optimal?

Answer: Consider a matrix with four cells: (a,a), (a,b), (b,a), and (b,b). Pareto optimal is a negative condition: Y is Pareto optimal to Z if Z is not better for all players (not if Y is better for all players). It may be that (a,a) is not worse than (a,b) and (a,b) is not worse than (b,b), but (a,a) is worse than (b,b).

In the prisoner’s dilemma, Jacob confess / Rachel confess is better for Jacob than Jacob not confess / Rachel confess, and Jacob not confess / Rachel confess is better for Rachel than Jacob not confess / Rachel not confess, but Jacob confess / Rachel confess is worse for both Jacob and Rachel than Jacob not confess / Rachel not confess.
For each set of outcomes in a two person game, know whether each person has a dominant strategy, which outcomes are Nash equilibria, and which outcomes are Pareto optimal.

The exercises here are taken from the Jack and Jill illustrations in the textbook. The final exam problems use other games with the same logic. Real games are more complex, since they involve many players, more than just two possible actions by each player, and mixed strategies.

Exercise 16.2: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays left or right; Rachel plays up or down.

<table>
<thead>
<tr>
<th>Rachel's Strategy</th>
<th>Jacob's Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>Jacob gets 1</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 1</td>
</tr>
<tr>
<td>Down</td>
<td>Jacob gets 2</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 4</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

Part A: Jacob’s dominant strategy is to play right, not left. Compare Jacob’s right vs left for each choice by Rachel.

- If Rachel plays up, Jacob’s right – left = 4 – 1 = +3.
- If Rachel plays down, Jacob’s right – left = 3 – 2 = +1.

Part B: Rachel’s dominant strategy is to play down, not up. Compare Rachel’s down vs up for each choice by Jacob.

- If Jacob plays left, Rachel’s down – up = 4 – 1 = +3.
- If Jacob plays right, Rachel’s down – up = 3 – 2 = +1.

Part C: If each player has a dominant strategy, one and only Nash equilibrium exists, and it is the combination of the two dominant strategies.

To verify, consider each cell of the game matrix.

- In the left / up cell, Jacob prefers to switch (and play right) and Rachel prefers to switch (and play down).
- In the left / down cell, Jacob prefers to switch (and play right).
- In the right / up cell, Rachel prefers to switch (and play down).
- In the right / down cell, neither Jacob nor Rachel wants to switch.

Part D: The game has three Pareto optimal outcomes: right / up, left / down, and right / down.

- Compared to right / up, Jacob loses in any other cell.
- Compared to left / down, Rachel loses in any other cell.
- Compared to right / down, either Jacob loses or Rachel loses in any other cell.

See Landsburg, *Price Theory*, Chapter 12, Game Theory, page 414
Exercise 16.3: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays left or right; Rachel plays up or down.

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>Jacob gets 1</td>
<td>Jacob gets 2</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 1</td>
<td>Rachel gets 4</td>
</tr>
<tr>
<td>Down</td>
<td>Jacob gets 4</td>
<td>Jacob gets 3</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 2</td>
<td>Rachel gets 3</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

Part A: Jacob does not have a dominant strategy. Compare Jacob’s right vs left for each choice by Rachel.

- If Rachel plays up, Jacob prefers to play right, since right – left = 2 – 1 = +1.
- If Rachel plays down, Jacob prefers to play left, since right – left = 3 – 4 = –1.

Part B: Rachel does not have a dominant strategy. Compare Rachel’s down vs up for each choice by Jacob.

- If Jacob plays left, Rachel prefers to play down, since down – up = 2 – 1 = +1.
- If Jacob plays right, Rachel prefers to play up, since down – up = 3 – 4 = –1.

Part C: If neither player has a dominant strategy, there may be zero, one, or two Nash equilibria.

Consider each cell of the game matrix.

- In the left / up cell, Jacob prefers to switch (and play right) and Rachel prefers to switch (and play down).
- In the left / down cell, neither Jacob nor Rachel wants to switch (= Nash equilibrium)
  - If Jacob switches to right, he loses $1.
  - If Rachel switches to up, she loses $1.
- In the right / up cell, neither Jacob nor Rachel wants to switch (= Nash equilibrium)
  - If Jacob switches to left, he loses $1.
  - If Rachel switches to down, she loses $1.
- In the right / down cell, Jacob prefers to switch (and play left) and Rachel prefers to switch (and play up).

Two of the outcomes are Nash equilibria.

Part D: The game has three Pareto optimal outcomes: right / up, left / down, and right / down.

- Compared to right / up, Rachel loses in any other cell.
- Compared to left / down, Jacob loses in any other cell.
- Compared to right / down, either Jacob loses or Rachel loses in any other cell.

See Landsburg, *Price Theory*, Chapter 12, Game Theory, page 414
Exercise 16.4: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays left or right; Rachel plays up or down.

<table>
<thead>
<tr>
<th></th>
<th><strong>Jacob’s Strategy</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Left</strong></td>
<td><strong>Right</strong></td>
</tr>
<tr>
<td><strong>Up</strong></td>
<td>Jacob gets 1</td>
<td>Jacob gets 4</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 1</td>
<td>Rachel gets 4</td>
</tr>
<tr>
<td><strong>Down</strong></td>
<td>Jacob gets 2</td>
<td>Jacob gets 3</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 2</td>
<td>Rachel gets 3</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

**Part A:** Jacob has a dominant strategy. Compare Jacob’s right vs left for each choice by Rachel.
- If Rachel plays up, Jacob prefers to play right, since right – left = 4 – 1 = +3.
- If Rachel plays down, Jacob prefers to play right, since right – left = 3 – 2 = +1.

**Part B:** Rachel does not have a dominant strategy. Compare Rachel’s down vs up for each choice by Jacob.
- If Jacob plays left, Rachel prefers to play down, since down – up = 2 – 1 = +1.
- If Jacob plays right, Rachel prefers to play down, since down – up = 3 – 4 = –1.

**Part C:** If neither player has a dominant strategy, there may be zero, one, or two Nash equilibria.

Consider each cell of the game matrix.
- In the left / up cell, Jacob prefers to switch (and play right) and Rachel prefers to switch (and play down).
- In the left / down cell, Jacob prefers to switch (and play right).
- In the right / up cell, neither Jacob nor Rachel wants to switch (= Nash equilibrium)
  - If Jacob switches to left, he loses $3.
  - If Rachel switches to down, she loses $1.
- In the right / down cell, Rachel prefers to switch (and play up).

One outcome is a Nash equilibria.

**Part D:** The game has one Pareto optimal outcomes: right / up.
- Compared to right / up, both Rachel and Jacob lose in any other cell.

See Landsburg, *Price Theory*, Chapter 12, Game Theory, page 414
Exercise 16.5: Game theory

Jacob and Rachel are players in a two-person game with the following outcomes. Jacob plays _left_ or _right_; Rachel plays _up_ or _down_.

<table>
<thead>
<tr>
<th></th>
<th>Jacob's Strategy</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Left</em></td>
<td><em>Right</em></td>
</tr>
<tr>
<td><em>Up</em></td>
<td>Jacob gets 2</td>
<td>Jacob gets 4</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 2</td>
<td>Rachel gets 1</td>
</tr>
<tr>
<td><em>Down</em></td>
<td>Jacob gets 1</td>
<td>Jacob gets 3</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 4</td>
<td>Rachel gets 3</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

**Part A:** Jacob’s dominant strategy is to play _right_, not _left_. Compare Jacob’s _right_ vs _left_ for each choice by Rachel.

- If Rachel plays _up_, Jacob’s _right_ – _left_ = 4 – 2 = +2.
- If Rachel plays _down_, Jacob’s _right_ – _left_ = 3 – 1 = +2.

**Part B:** Rachel’s dominant strategy is to play _down_, not _up_. Compare Rachel’s _down_ vs _up_ for each choice by Jacob.

- If Jacob plays _left_, Rachel’s _down_ – _up_ = 4 – 2 = +2.
- If Jacob plays _right_, Rachel’s _down_ – _up_ = 3 – 1 = +2.

**Part C:** If each player has a dominant strategy, one and only Nash equilibrium exists, and it is the combination of the two dominant strategies.

To verify, consider each cell of the game matrix.

- In the _left_ / _up_ cell, Jacob prefers to switch (and play _right_) and Rachel prefers to switch (and play _down_).
- In the _left_ / _down_ cell, Jacob prefers to switch (and play _right_).
- In the _right_ / _up_ cell, Rachel prefers to switch (and play _down_).
- In the _right_ / _down_ cell, neither Jacob nor Rachel wants to switch.

**Part D:** The game has three Pareto optimal outcomes: _right_ / _up_, _left_ / _down_, and _right_ / _down_.

- Compared to _right_ / _up_, Jacob loses in any other cell.
- Compared to _left_ / _down_, Rachel loses in any other cell.
- Compared to _right_ / _down_, either Jacob loses or Rachel loses in any other cell.

Exercise 16.6: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays \textit{left} or \textit{right}; Rachel plays \textit{up} or \textit{down}.

<table>
<thead>
<tr>
<th>Rachel's Strategy</th>
<th>Jacob's Strategy</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>Jacob gets 1</td>
<td>Jacob gets 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rachel gets 3</td>
<td>Rachel gets 1</td>
<td></td>
</tr>
<tr>
<td>Down</td>
<td>Jacob gets 4</td>
<td>Jacob gets 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rachel gets 2</td>
<td>Rachel gets 4</td>
<td></td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

\textbf{Part A:} Jacob does not have a dominant strategy. Compare Jacob's \textit{right} vs \textit{left} for each choice by Rachel.

- If Rachel plays \textit{up}, Jacob's $\textit{right} - \textit{left} = 3 - 1 = +2$.
- If Rachel plays \textit{down}, Jacob's $\textit{right} - \textit{left} = 2 - 4 = -2$.

\textbf{Part B:} Rachel does not have a dominant strategy. Compare Rachel's \textit{down} vs \textit{up} for each choice by Jacob.

- If Jacob plays \textit{left}, Rachel's $\textit{down} - \textit{up} = 2 - 3 = -1$.
- If Jacob plays \textit{right}, Rachel's $\textit{down} - \textit{up} = 4 - 1 = +3$.

\textbf{Part C:} If neither player has a dominant strategy, there may be zero, one, or two Nash equilibria.

Consider each cell of the game matrix.

- In the \textit{left} / \textit{up} cell, Jacob prefers to switch (and play \textit{right}).
- In the \textit{left} / \textit{down} cell, Rachel prefers to switch (and play \textit{up}).
- In the \textit{right} / \textit{up} cell, Rachel prefers to switch (and play \textit{down}).
- In the \textit{right} / \textit{down} cell, Jacob prefers to switch (and play \textit{left}).

No outcome is a Nash equilibria.

\textbf{Part D:} The game has two Pareto optimal outcomes: \textit{left} / \textit{down} and \textit{right} / \textit{down}.

- Compared to \textit{right} / \textit{up}, both Jacob and Rachel gain in \textit{left} / \textit{down}.
- Compared to \textit{left} / \textit{up}, both Jacob and Rachel gain in \textit{right} / \textit{down}.
- Compared to \textit{left} / \textit{down}, Jacob loses in any other cell.
- Compared to \textit{right} / \textit{down}, Rachel loses in any other cell.

See Landsburg, \textit{Price Theory}, Chapter 12, Game Theory, page 414.
Exercise 16.7: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays left or right; Rachel plays up or down.

<table>
<thead>
<tr>
<th></th>
<th>Jacob's Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>Jacob gets 2</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 2</td>
</tr>
<tr>
<td>Down</td>
<td>Jacob gets 1</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 1</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

Part A: Jacob does not have a dominant strategy. Compare Jacob’s right vs left for each choice by Rachel.

- If Rachel plays up, Jacob’s right – left = 1 – 2 = –1.
- If Rachel plays down, Jacob’s right – left = 3 – 1 = +2.

Part B: Rachel does not have a dominant strategy. Compare Rachel’s down vs up for each choice by Jacob.

- If Jacob plays left, Rachel’s down – up = 1 – 2 = –1.
- If Jacob plays right, Rachel’s down – up = 3 – 1 = +2.

Part C: If neither player has a dominant strategy, there may be zero, one, or two Nash equilibria.

Consider each cell of the game matrix.

- In the left / up cell, neither Jacob nor Rachel prefers to switch.
- In the left / down cell, Rachel prefers to switch (and play up) and Jacob prefers to switch (and play right).
- In the right / up cell, Rachel prefers to switch (and play down) and Jacob prefers to switch (and play left).
- In the right / down cell, neither Jacob nor Rachel prefers to switch.

Two outcomes are Nash equilibria.

Part D: The game has one Pareto optimal outcome: right / down, which is best for both Jacob and Rachel.

See Landsburg, *Price Theory*, Chapter 12, Game Theory, page 414
Exercise 16.8: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays left or right; Rachel plays up or down.

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>Jacob gets 2</td>
<td>Jacob gets 1</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 3</td>
<td>Rachel gets 1</td>
</tr>
<tr>
<td>Down</td>
<td>Jacob gets 1</td>
<td>Jacob gets 3</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 1</td>
<td>Rachel gets 2</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

Part A: Jacob does not have a dominant strategy. Compare Jacob’s right vs left for each choice by Rachel.

- If Rachel plays up, Jacob’s right – left = 1 – 2 = −1.
- If Rachel plays down, Jacob’s right – left = 3 – 1 = +2.

Part B: Rachel does not have a dominant strategy. Compare Rachel’s down vs up for each choice by Jacob.

- If Jacob plays left, Rachel’s down – up = 1 – 3 = −2.
- If Jacob plays right, Rachel’s down – up = 2 – 1 = +1.

Part C: If neither player has a dominant strategy, there may be zero, one, or two Nash equilibria.

Consider each cell of the game matrix.

- In the left / up cell, neither Jacob nor Rachel prefers to switch.
- In the left / down cell, Rachel prefers to switch (and play up) and Jacob prefers to switch (and play right).
- In the right / up cell, Rachel prefers to switch (and play down) and Jacob prefers to switch (and play left).
- In the right / down cell, neither Jacob nor Rachel prefers to switch.

Two outcomes are Nash equilibria.

Part D: The game has two Pareto optimal outcomes: left / up and right / down.

- Compared to left / up, Rachel loses in any other cell.
- Compared to right / down, Jacob loses in any other cell.

See Landsburg, Price Theory, Chapter 12, Game Theory, page 414
Exercise 16.9: Game theory

Jacob and Rachel are players in a two person game with the following outcomes. Jacob plays left or right; Rachel plays up or down.

<table>
<thead>
<tr>
<th></th>
<th>Jacob's Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Up</td>
<td>Jacob gets 12</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 8</td>
</tr>
<tr>
<td>Down</td>
<td>Jacob gets 15</td>
</tr>
<tr>
<td></td>
<td>Rachel gets 7</td>
</tr>
</tbody>
</table>

A. Does Jacob have a dominant strategy?
B. Does Rachel have a dominant strategy?
C. How many Nash equilibria does the game have?
D. How many Pareto optimal outcomes does the game have?

Part A: Jacob’s dominant strategy is to play left, not right. Compare Jacob’s right vs left for each choice by Rachel.
- If Rachel plays up, Jacob’s right – left = 9 – 12 = −3.
- If Rachel plays down, Jacob’s right – left = 14 – 15 = −1.

Part B: Rachel does not have a dominant strategy. Compare Rachel’s down vs up for each choice by Jacob.
- If Jacob plays left, Rachel’s down – up = 7 – 8 = −1.
- If Jacob plays right, Rachel’s down – up = 10 – 8 = +2.

Part C: If only one player has a dominant strategy, there may be zero, one, or two Nash equilibria.

Consider each cell of the game matrix.
- In the left / up cell, neither Jacob nor Rachel prefers to switch; this is a Nash equilibrium.
- In the left / down cell, Rachel prefers to switch (and play up).
- In the right / up cell, Jacob prefers to switch (and play left).
- In the right / down cell, Jacob prefers to switch (and play left).

One outcome is a Nash equilibrium.

Part D: The game has two Pareto optimal outcomes: left / down and right / down.
- Compared to left / down, Jacob loses in any other cell.
- Compared to right / down, Rachel loses in any other cell.
- left / up and right / up are both worse than right / down for both Jacob and Rachel.

See Landsburg, Price Theory, Chapter 12, Game Theory, page 414